

Gauge Theories in the Momentum/Curvature Representation

R. Jackiw*

Center for Theoretical Physics

Massachusetts Institute of Technology

Cambridge, MA 02139-4307

American Mathematical Society, New York NY, April 1996

*This work is supported in part by funds provided by the U.S. Department of Energy (D.O.E.)
under contract #DE-FC02-94ER40818. MIT-CTP-2516 March 1996

I. INTRODUCTION

I shall discuss some kinematical properties of non-Abelian quantum gauge theories that are elementary, but not widely appreciated. Use will be made of various nice mathematical structures, so I hope the material will interest this audience.

Consider a generic non-abelian gauge theory, with action presented in first order form.

$$I = \int dt \int d^d r \left[E_a^i \dot{A}_i^a - \mathcal{H}(E, A) + A_0^a G_a \right] \quad (1)$$

Here the covariant spatial components of the gauge potential (connection) A_i^a are the canonical coordinates and the conjugate momenta are identified from the first term in the integrand (1) as E_a^i . Indeed the first term is the (functional) canonical 1-form, analogous to the 1-form arising in particle mechanics: $\int dt p \dot{q} = \int p dq$ (overdot indicates time-differentiation). Further, in (1) \mathcal{H} denotes the Hamiltonian density, and G_a is the Gauss-law generator whose vanishing is enforced by the Lagrange multiplier A_0^a , which is also the temporal component of the gauge potential. The dimensionality d of space, over whose volume the spatial integral is taken, has been left unspecified; also unspecified is the explicit form for \mathcal{H} . However we assume that G_a generates the usual gauge transformation on A_i^a with parameters θ^a and structure constants $f_{bc}^a = -f_{cb}^a$,

$$\delta_\theta A_i^a = \partial_i \theta^a + f_{bc}^a A_i^b \theta^c \quad (2)$$

Further, we assume that E_a^i transforms covariantly.

$$\delta_\theta E_a^i = -f_{ab}^c \theta^b E_c^i \quad (3)$$

Various gauge field models fit our requirements, but not theories with a Chern-Simons term: for these the canonical momentum does not transform covariantly. The usual Yang-Mills model in any spatial dimension satisfies the desired requirements. Its dynamics, conventionally derived from the second-order Lagrange density

$$\mathcal{L}_{YM} = -\frac{1}{4} F_{\mu\nu}^a F^{\mu\nu a}$$

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + f_{bc}^a A_\mu^b A_\nu^c \quad (4a)$$

can be equivalently encoded in the first order action (1), with

$$\mathcal{H} = \frac{1}{2}E_a^i E_a^i + \frac{1}{4}F_{ij}^a F^{ij a} \quad (4b)$$

and

$$G_a = (D_i E^i)_a = \partial_i E_a^i + f_{ab}{}^c A_i^b E_c^i \quad (4c)$$

where the canonical momentum E_a^i coincides with the non-Abelian electric field (curvature) F_{oi}^a . For $d = 1$, there is no magnetic contribution to \mathcal{H} , since F_{ij}^a does not exist. Additionally, in one spatial dimension, another gauge theory may be constructed: the so-called “ B - F ” model, described by the covariant Lagrange density

$$\mathcal{L}_{BF} = \frac{1}{2}\eta_a \varepsilon^{\mu\nu} F_{\mu\nu}^a \quad (5a)$$

that is already in first order form, and so is governed by the action (1), with vanishing \mathcal{H} , and

$$G_a = (D\eta)_a = \eta'_a + f_{ab}{}^c A_1^b \eta_c \quad (5b)$$

(Dash indicates differentiation with respect to the single spatial coordinate x ; $\varepsilon^{\mu\nu}$ is the two-dimensional anti-symmetric tensor.) The gauge covariant η_a is identified with E_a^1 . (The B in the “ B - F ” nomenclature refers to $B_a^{\mu\nu} \equiv \frac{1}{2}\eta_a \varepsilon^{\mu\nu}$.) Note that owing to the vanishing of the “ B - F ” Hamiltonian, the problem of solving a “ B - F ” quantum theory reduces to solving its Gauss Law, *i.e.* finding states annihilated by G_a of (5b).

In the quantized theory, the canonical variables satisfy equal-time commutation relations

$$i[E_a^i(\mathbf{r}), A_j^b(\mathbf{r}')] = \delta_j^i \delta_a^b \delta(\mathbf{r} - \mathbf{r}') \quad (6)$$

and the gauge transformation rules (2), (3) are gotten by commuting with $\int d^d r \theta_a G^a \equiv G_\theta$.

$$[G_\theta, A_i^a] = i\delta_\theta A_i^a \quad (7a)$$

$$[G_\theta, E_a^i] = i\delta_\theta E_a^i \quad (7b)$$

(A common time argument in all operators is suppressed.)

An explicit realization is given in a Schrödinger representation, where states are described by wave functionals of A , $\Psi(A)$, and the action of the operator A is realized by multiplication by A , while E is realized by functional differentiation: $E \sim \frac{1}{i} \frac{\delta}{\delta A}$. Moreover, physical states are annihilated by G_a , which also means that the wave functionals are gauge invariant

$$\Psi(A_i^U) \equiv \Psi(U^{-1}A_iU + U^{-1}\partial_iU) = \Psi(A_i) \quad (8)$$

(Frequently we use group-index free notation: $A_i \equiv A_i^a T_a$, *etc.*, where T_a are anti-Hermitian Lie algebra generators; also $\langle A, E \rangle \equiv A_i^a E_a^i$.) For $d = 3$, where the gauge transformation U can be homotopically non-trivial, a phase involving the vacuum angle may arise in the response of the wave functional to a gauge transformation; here I shall ignore this complication.

The realization described above is the field theoretical analog of the quantum mechanical story, where wave functions depend on q , $\psi(q)$, the operator q acts by multiplication and p is realized as a derivative $\frac{1}{i} \frac{d}{dq}$. But in quantum mechanics, we may also use the momentum representation, where p acts by multiplication on wave functions that depend on p , $\varphi(p)$, and q is realized by differentiation $i \frac{d}{dp}$. The relation between the two is given by a Fourier transformation.

$$\varphi(p) = \int \frac{dq}{\sqrt{2\pi}} e^{-ipq} \psi(q) \quad (9)$$

I shall discuss here some properties of the field theoretic momentum representation, where E acts by multiplication on wave functionals that depend on E , while A is realized by (functional) differentiation as $i \frac{\delta}{\delta E}$.¹

II. RESPONSE TO GAUGE TRANSFORMATIONS

While physical states in the “ A ” representation are gauge invariant, see Eq. (8), those in the “ E ” representation are not. This is immediately established by using the (functional) Fourier transform relation between functionals $\Phi(E)$ in the “ E ” representation, and the gauge invariant wave functionals $\Psi(A)$ of the “ A ” representation. The following chain of equations holds.

$$\Phi(E) = \int \mathcal{D} A \left(\exp -i \int d^d r \langle E, A \rangle \right) \Psi(A)$$

$$\begin{aligned}
&= \int \mathcal{D} A \left(\exp -i \int d^d r \langle E, A \rangle \right) \Psi(U^{-1} A U + U^{-1} \partial U) \\
&= \int \mathcal{D} A \left(\exp -i \int d^d r \langle E^U, A \rangle \right) \Psi(A + U^{-1} \partial U) \\
&= \exp i \int d^d r \langle E, \partial U U^{-1} \rangle \int \mathcal{D} A \left(\exp -i \int d^d r \langle E^U, A \rangle \right) \Psi(A) \\
&= \exp -i \Omega(E, U) \Phi(E^U)
\end{aligned} \tag{10}$$

The first equation is the field theoretic analog to (9). The second equality is true because $\Psi(A)$ is gauge invariant. In the third equality we have changed integration variables: $A \rightarrow U A U^{-1}$; this has unit Jacobian, and affects the phase by replacing E with its gauge transform $E^U = U^{-1} E U$. In the next step, A_i is shifted: $A_i \rightarrow A_i - U^{-1} \partial_i U$; this produces the phase $\Omega(E, U)$ seen in the last equality.

$$\Omega(E, U) = - \int d^d r E_a^i (\partial_i U U^{-1})^a \tag{11}$$

Thus from (10), it follows that physical wave functionals in the “ E ” representation are **not** gauge invariant. Rather, after a gauge transformation they acquire the phase $\Omega(E, U)$,

$$\Phi(E^U) = e^{i \Omega(E, U)} \Phi(E) \tag{12}$$

which is recognized to be a 1-cocycle *i.e.* $\Omega(E, U)$ satisfies

$$\Omega(E, U_1 U_2) = \Omega(E^{U_1}, U_2) + \Omega(E, U_1) \tag{13}$$

as is required by (12) when two gauge transformations are composed.

We conclude therefore that physical functionals in the “ E ” representation, which are annihilated by the Gauss law generator G_a , obey (12). Before exploring further properties of that equation, let us give another perspective on the result. [1]

III. GEOMETRIC QUANTIZATION

One may pose the following question: why is it that functionals of the gauge covariant variable E are not gauge invariant, while functionals of the gauge non-invariant variable A are gauge invariant. The answer lies in the fact that the canonical 1-form $\int dt \int d^d r E_a^i \dot{A}_i^a$

is not gauge invariant. The best way to understand this statement is in the context of geometric quantization.² So I shall first briefly review that formalism, using for simplicity ordinary quantum mechanics as an illustration.

Collect the canonical variables p, q into the pair $\xi^m : \xi^1 = p, \xi^2 = q$, which serve as coordinates for the two-dimensional phase space. The canonical 1-form pdq is written as

$$\begin{aligned}\theta &= \theta_m d\xi^m \\ \theta_1 &= 0, \theta_2 = p\end{aligned}\tag{14}$$

while the symplectic 2-form reads

$$\begin{aligned}\omega &= d\theta = \frac{1}{2}\omega_{mn}d\xi^m d\xi^n \\ \omega_{mn} &= \frac{\partial\theta_n}{\partial\xi^m} - \frac{\partial\theta_m}{\partial\xi^n} = \varepsilon_{mn}\end{aligned}\tag{15}$$

Canonical transformations are coordinate transformations on phase space that leave ω invariant; infinitesimally they are given by a vector field $v^m(\xi)$.

$$\delta\xi^m = -v^m(\xi)\tag{16}$$

A “generator” $G(\xi)$ for a vector field v^m is defined by

$$v^m\omega_{mn} = -\frac{\partial G}{\partial\xi^n}\tag{17}$$

Conversely, for any function $G(\xi)$ on phase space, we can use (17) to define a vector field v^m . (It is assumed that ω_{mn} is non-degenerate, *i.e.* it has an inverse ω^{mn} ; in our case $\omega^{mn} = -\varepsilon^{mn}$.)

Within geometric quantization, there is a stage called “pre-quantization” that arises before the conventional quantum theory is defined. One works with pre-quantized wave functions $f(\xi)$ that vary over the entire phase space, *i.e.* they depend on **both** p and q . To every quantity $G(\xi)$ one associates a pre-quantized operator \hat{G} that acts on the $f(\xi)$. The operator is given by

$$\hat{G} = \frac{1}{i}v^m\mathcal{D}_m + G\tag{18}$$

where v^m is defined from G by (17), and \mathcal{D}_m is the covariant derivative

$$\mathcal{D}_m \equiv \frac{\partial}{\partial \xi^m} - i\theta_m \quad (19)$$

Thus the coordinate $q = \xi^2$ produces, according to (15) and (17), the vector field $v^m = (-1, 0)$ and according to (14), (18) and (19), the pre-quantized operator is

$$\hat{q} = \frac{1}{i}v^m\mathcal{D}_m + q = i\frac{\partial}{\partial p} + \theta_1 + q = i\frac{\partial}{\partial p} + q \quad (20)$$

Similarly, $p = \xi^1$ leads to $v^m = (0, 1)$, and

$$\hat{p} = \frac{1}{i}v^m\mathcal{D}_m + p = \frac{1}{i}\frac{\partial}{\partial q} - \theta_2 + p = \frac{1}{i}\frac{\partial}{\partial q} \quad (21)$$

Finally the quantum theory, with its Hilbert space, is defined by choosing a “polarization”. This consists of fixing polarization vector fields π^m , which span **half** the (even-dimensional) phase space, and imposing on the pre-quantized functions $f(\xi)$ the conditions

$$\pi^m\mathcal{D}_m f = 0 \quad (22)$$

Equations (22) determine dependence on half the phase-space coordinates, leaving arbitrary the dependence on the other half, and quantum mechanical wave functions are solutions to (22).

Selection of the conventional “coordinate” representation is accomplished by using the vector field corresponding to $q = \xi^2$: $\pi^m = (-1, 0)$. Wave functions in the coordinate polarization therefore satisfy

$$\mathcal{D}_1 f_{\text{coordinate}} = \frac{\partial}{\partial p} f_{\text{coordinate}} = 0 \quad (23a)$$

which is solved by arbitrary functions of q that become the quantum mechanical wave functions.

$$f_{\text{coordinate}} = \psi(q) \quad (23b)$$

The operators \hat{q} and \hat{p} , whose form is given in (20) and (21), act as expected.

$$\hat{q}f_{\text{coordinate}} = q\psi(q)$$

$$\hat{p}f_{\text{coordinate}} = \frac{1}{i} \frac{d}{dq} \psi(q) \quad (24)$$

The alternative “momentum” polarization uses the vector field corresponding to $p = \xi^1 : \pi^m = (0, 1)$. The polarization condition becomes

$$\mathcal{D}_2 f_{\text{momentum}} = \left(\frac{\partial}{\partial q} - ip \right) f_{\text{momentum}} = 0 \quad (25a)$$

which is solved by arbitrary functions of p , times a phase involving q

$$f_{\text{momentum}} = e^{ipq} \varphi(p) \quad (25b)$$

For the quantum mechanical wave function, the phase is stripped away from the pre-quantized function, leaving $\varphi(p)$. Action of operators \hat{q} and \hat{p} on φ is deduced from their action on f .

$$\hat{q}f_{\text{momentum}} = \left(i \frac{\partial}{\partial p} + q \right) e^{ipq} \varphi(p) = e^{ipq} i \frac{d}{dp} \varphi(p) \quad (26a)$$

$$\hat{p}f_{\text{momentum}} = \frac{1}{i} \frac{\partial}{\partial q} e^{ipq} \varphi(p) = e^{ipq} p \varphi(p) \quad (26b)$$

Once again the expected formulas emerge in the action on $\varphi(p)$.

We are ready now to examine our gauge theory within the above formalism. Since the symplectic 1-form $\int dt \int d^d r E_a^i \dot{A}_i^a$ is the field theoretic generalization of the particle expression $\int p dq$, we may immediately take over the previous results, with field variables replacing particle variables ($p \rightarrow E, q \rightarrow A$) in a pre-quantized wave functional depending on E and A , $F(E, A)$.

Next we determine the pre-quantized operator that corresponds to G_θ . A straight forward calculation shows that

$$\hat{G}_\theta = i \int d^d r \left(\delta_\theta E_a^i \frac{\delta}{\delta E_a^i} + \delta_\theta A_i^a \frac{\delta}{\delta A_i^a} \right) \quad (27)$$

i.e. \hat{G}_θ effects an infinitesimal gauge transformation on the pre-quantized wave functional $F(E, A)$. Moreover, demanding that \hat{G}_θ annihilate F , thereby imposing Gauss’ law at the pre-quantized level, ensures that F is gauge invariant.

But now recall that in the coordinate polarization the pre-quantized wave functional, restricted to depend solely on A , coincides with the quantized wave functional $\Psi(A)$.

$$F_{\text{coordinate}} = \Psi(A) \quad (28)$$

Therefore $\Psi(A)$ is per force gauge invariant. On the other hand, with momentum polarization the gauge invariant pre-quantized functional is given by

$$F_{\text{momentum}} = e^{i \int d^d r \langle E, A \rangle} \Phi(E) \quad (29)$$

Hence gauge invariance of F requires

$$e^{i \int d^d r \langle E^U, A^U \rangle} \Phi(E^U) = e^{i \int d^d r \langle E, A \rangle} \Phi(E) \quad (30)$$

This then is equivalent to (12).

To summarize: in geometric quantization, the pre-quantized wave functional is gauge invariant, and so is the quantum wave functional in the coordinate polarization. But in the momentum polarization, owing to the gauge non-invariance of the canonical 1-form, the quantum wave functional is not gauge invariant.

IV. PROPERTIES OF THE COCYCLE AND WAVE FUNCTIONAL

From the gauge transformation law (12) for the wave functional Φ , we can deduce some of Φ 's properties. Of course no gauge invariant portion of Φ is affected by (12); so no information will be forthcoming on this aspect of the wave functional.

It must be emphasized that non-trivial information is available only in the non-Abelian case. For an Abelian theory, with gauge invariant E and $U = e^{i\theta}$, where θ is function (not a matrix), it follows from (11) that $\Omega(E, U) = \int d^d r \partial_i E^i \theta$ and (12) or (30) merely require that Φ have support only on the transverse part of E^i . This is the momentum-space analog of the position-space condition that $\Psi(A)$ in the Abelian theory has support only on the transverse (gauge invariant) portion of A_i .

Returning now to the non-Abelian case, we extract a gauge non-invariant eikonal factor from the wave functional, leaving a gauge invariant functional $\hat{\Phi}(E)$.

$$\Phi(E) = e^{iS(E)} \hat{\Phi}(E) \quad (31a)$$

$$\hat{\Phi}(E^U) = \hat{\Phi}(E) \quad (31b)$$

Note that the gauge invariant functional $\hat{\Phi}(E)$ is annihilated by the “rotation” part of the Gauss generator $i f_{ab}{}^c E_c^i A_i^b \hat{\Phi}(E) = i f_{ab}{}^c E_c^i \frac{\delta}{\delta E_a^i} \hat{\Phi}(E) = 0$; $\hat{\Phi}(E)$ is **not** annihilated by the full Gauss generator owing to its $\partial_i E_a^i$ part. (Thus we see again that physical wave functionals in the “ E ” representation cannot be gauge invariant, because gauge invariant functionals are **not** annihilated by the Gauss generator.) From (12) and (31) it follows that $S(E)$ must satisfy

$$S(E^U) - S(E) = - \int d^d r E_a^i (\partial_i U U^{-1})^a \quad (32)$$

[An integer multiple of (2π) can also be present in (32).] This formula would indicate that the 1-cocycle is trivial, since it appears expressible as a coboundary, *i.e.* as the difference on the left side of (32). However, such a conclusion would be misleading because $S(E)$ is necessarily singular. To see that, present (32) infinitesimally as

$$S(E + [E, \theta]) - S(E) = \int d^d r \partial_i E_a^i \theta^a \quad (33)$$

If S is a non-singular functional of E^i , we can choose E^i so that it commutes with θ in the Lie algebra, whereupon the left side vanishes, while the right side need not. The contradiction is resolved by allowing $S(E)$ to possess singularities, see below.

Before attempting to solve for $S(E)$ from (32), let us observe that (32) also implies that the quantity

$$\mathcal{A}_i^a(E) \equiv - \frac{\delta S(E)}{\delta E_a^i} \quad (34)$$

transforms as a gauge connection

$$\mathcal{A}_i(E^U) = U^{-1} \mathcal{A}_i(E) U + U^{-1} \partial_i U \quad (35)$$

This suggests that a formula for $S(E)$ could have the form

$$S(E) = - \int d^d r E_a^i (g^{-1} \partial_i g)^a \quad (36)$$

where g is an as-yet-to-be-determined functional of E , with the property that it transforms like a group element.

$$g(E^U) = g(E)U \quad (37)$$

The gauge connection (34) becomes

$$\mathcal{A}_i^a = (g^{-1}\partial_i g)^a - \int d^d r \partial_j (g E^j g^{-1})_b \left(\frac{\delta g}{\delta E_a^i} g^{-1} \right)^b \quad (38)$$

We now discuss separately the one-dimensional ($d = 1$) and the higher-dimensional ($d \geq 2$) models.

A. One Dimension

In one dimension, the Gauss law reads

$$\left(\eta'_a + f_{ab}{}^c \eta_c^i \frac{\delta}{\delta \eta_b^i} \right) \Phi(\eta) = 0 \quad (39)$$

(We have replaced E_a^1 by η_a .) Contracting this equation with η_a and using anti-symmetry of the structure constants to eliminate the second term shows that $\Phi(\eta)$ has support only on vanishing $(\eta_a \eta_a)'$, *i.e.* on η fields that are in the orbit of a constant. Consequently in (38) we can choose g to be that group element which takes η to the constant, so that the last term is absent.

$$\eta = g^{-1} K g$$

$$K \text{ constant and invariant} \quad (40)$$

It then follows that S may be written as

$$S(\eta) = - \int dx K_a \left(\frac{dg}{dx} g^{-1} \right)^a \quad (41)$$

with g related to η by (40). The transformation law (32) is straightforwardly verified from (37) and (40). Note that the connection (34) becomes a pure gauge. [4], [5]

The above structure (41) has another role in mathematical physics, quite distinct from the role in which we encounter it here as the phase of a wave functional. Observe that S

in (41) is given by an integral of the 1-form $\langle K, dgg^{-1} \rangle$, which one may take as a canonical 1-form for a Lagrangian with variables depending on “time”. It is then further true that the symplectic 2-form, $d\langle K, dgg^{-1} \rangle = \langle K, dgg^{-1}dgg^{-1} \rangle$ defines Poisson brackets and that the brackets of the quantities $Q^a = (g^{-1}Kg)^a$ reproduce the Lie algebra of the relevant group. This 2-form is associated with the names Kirillov and Kostant.³ [One recognizes here a development that has previously occurred in connection with the Chern-Simons term: this term first arose in physics as the phase of the QCD wave functional in (3+1)-dimensional Yang-Mills theory, whose gauge transformation response gives rise to the vacuum angle. Subsequently the Chern-Simons term was used in dynamics for a lower-dimensional field theory.⁴]

Formula (41) can be presented for any group, but it is not explicit, in that the group element’s dependence on η is defined only implicitly by (40). For a specific Lie group, an explicit formula may be given by expressing g in terms of η . For example for $SU(2)$ one finds

$$S(\eta) = \frac{1}{3!} \int dx \varepsilon^{abc} \eta'_a \tan^{-1} \frac{\eta_b}{\eta_c}$$

$$\eta_a \eta_a \text{ constant} \tag{42}$$

This expression also puts into evidence the singularities of S that were mentioned earlier. Finally we remark that since the “ B - F ” Hamiltonian vanishes, the entire problem of quantization, which reduces to satisfying the Gauss law, is solved by the wave functionals (31a), with S given by (41).

B. Higher Dimensions

In higher dimensions there does not appear to be a general, mathematically elegant, formula for $S(E)$ valid for arbitrary groups. Specific expressions can be given, and for $SU(2)$ one has [3]

$$S(E) = \frac{1}{d-1} \int d^d r \varepsilon^{abc} (E_a^i E_b^j \partial_i E_c^k) \phi_{jk} \tag{43}$$

where the gauge invariant ϕ_{jk} is defined by

$$E_a^i E_a^j \phi_{jk} = \delta_k^i \quad (44)$$

and exhibits the singularities that necessarily are present in (43). With (43), the connection (34) is no longer a pure gauge; it gives rise to non-vanishing curvature.

V. APPLICATIONS

The above ideas may have application in analyzing gauge theories – the Yang-Mills models in spatial dimension greater than one, as well as “ B - F ” theories, which are defined on a line and have been used to model lineal gravity.

A. Yang-Mills Theories

In the Yang-Mills case, the above line of research is motivated by the expectation that the mysteries of non-Abelian gauge theories at low energy – like, for example, confinement and the spectrum of low-lying states – can be unraveled when gauge covariant variables – like E – are used. In this connection, the singularities of $S(E)$ are viewed as analogous to the centrifugal barrier that is seen in particle quantum mechanics, when radial (rotation covariant) variables are used. It is hoped that analysis of these singularities will provide clues to low energy dynamics – but it is also true that thus far the hope has not been fulfilled.

B. Gravity Theories

In the one-dimensional case, the “ B - F ” theory arises in gauge theoretic reformulations of lineal gravity. This comes about in the following fashion.

If one wants to construct a gravitational theory on a line, *i.e.* in (1+1)-dimensional space-time, Einstein’s general relativity model cannot be used, because the Einstein tensor (which enters the general relativistic field equation) vanishes identically, since in two dimensions, the Ricci tensor $R_{\mu\nu}$ is proportional to the scalar R : $R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 0$. Correspondingly the Einstein-Hilbert Lagrange density $\sqrt{-g}R$ is a total derivative, and does not give rise to Euler-Lagrange equations.

A way around this impasse was suggested some years ago. One introduces an additional non-geometrical, world scalar variable η and uses, instead of the Einstein-Hilbert formula, the Lagrange density [9]

$$\mathcal{L} = \sqrt{-g}\eta R + \dots \quad (45)$$

where the ellipsis stands for further η - and metric-dependent terms, which give rise to different theories. More recently models of this type have been abstracted from string theory, and in this context they are called “dilaton-gravity” theories, with $\ln\eta$ being identified as the “dilaton” field. Alternatively, they are also known as “scalar-tensor theories”, η being the scalar and $g_{\mu\nu}$ the tensor.

It turns out that several specific scalar-tensor models, with specific expressions for the ellipsis in the above Lagrange density, can be equivalently formulated as gauge theories of the “ B - F ” variety. Such formulations proceed along the following steps. [10], [11]

Step 1. For gravitational variables do **not** use the metric tensor, but rather the Einstein-Cartan variables: the *Vielbein* e_μ^a and the spin-connection ω_μ^{ab} . Here, as before, Greek letters index space-time components, while Roman letters denote components in a flat tangent space, with metric η_{ab} . [Note that the present tangent space index “a” does not have the same meaning as in Sections I-IV, where it ranged over the Lie algebra.] The metric tensor is given by

$$g_{\mu\nu} = e_\mu^a e_\nu^b \eta_{ab} \quad (46)$$

In the two-dimensional application, we may set $\omega_\mu^{ab} = \varepsilon^{ab}\omega_\mu$, and $\eta_{ab} = \text{diag}(1, -1)$. In addition to e_μ^a and ω_μ , it may be necessary to use further variables, see below. At this stage one has in hand a gauge theory of the local Lorentz group, which in two space-time dimensions contains the single generator J , and ω_μ is the associated gauge potential. The *Zweibein* e_μ^a transforms covariantly under the Lorentz group – it is not a potential.

Step 2. To have a completely gauge theoretic description of the gravity theory, we consider translations, generated by P_a , and take the *Zweibeine* to be the associated gauge potentials.

Step 3. To close the algebraic system, we look to the Lie algebra of the generators J and P_a . As is conventional, we let J generate rotations on P_a

$$[P_a, J] = \varepsilon_a{}^b P_b \quad (47)$$

But for the $[P_a, P_b]$ commutator we have a choice: (1) it can vanish – as in the three-parameter Poincaré group; (2) it can be proportional to J – as in the three-parameter DeSitter or anti-DeSitter groups; (3) it can close on a central element that commutes with P_a and J – as in the centrally extended Poincaré group, and this option is available only in two dimensions. So we take for the first two choices

$$[P_a, P_b] = \varepsilon_{ab} \lambda J \quad (48a)$$

or for the third choice

$$[P_a, P_b] = \varepsilon_{ab} I \quad (48b)$$

In Eq. (48a), choice (1) above is realized in the limit $\lambda \rightarrow 0$. In Eq. (48b), I is the central element. That quantity is taken as an additional generator, commuting with J and P_a , so the centrally extended Poincaré group is viewed as a four-parameter group. Consequently, if Eq. (48b) is chosen, a further gauge potential must supplement ω_μ and e_μ^a , we call it a_μ and associate it with I .

Step 4. A Lie-algebra valued connection is constructed as

$$A_\mu = e_\mu^a P_a + \omega_\mu J + a_\mu I \quad (49)$$

The curvature is constructed by the usual formula

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \quad (50a)$$

where the commutator is evaluated from (47) and (48). Eq. (50a) defines the curvature components

$$F_{\mu\nu} = f_{\mu\nu}^a P_a + f_{\mu\nu} J + a_{\mu\nu} I \quad (50b)$$

which can then be used in a “ B - F ” Lagrange density

$$\mathcal{L}_{BF} = \frac{\varepsilon^{\mu\nu}}{2} \left(\eta_a f_{\mu\nu}^a + \eta_2 f_{\mu\nu} + \eta_3 a_{\mu\nu} \right) \quad (51)$$

The last term in (49), (50b) and (51), referring to the central direction in the Lie algebra, is present only when (48a) is used. In this way the Lagrange density of (5a) arises in the description of lineal gravity. [Note that the index “ a ” in the previous Sections, in particular in (5a), ranges over the entire Lie algebra, while in the present sub-Section it denotes the two tangent space components, $a = \{0, 1\}$].

The dynamics based on (51) entails the requirement that the covariant derivative of the Lagrange multiplier multiplet (η_a, η_2, η_3) vanishes (this is obtained by varying A_μ), and the condition that the curvature $F_{\mu\nu}$ vanishes (this is obtained by varying the Lagrange multipliers.).

One then shows that if the chosen Lie algebra is (48a), the above dynamics is equivalent to that of the first-positated scalar-tensor gravity theory, which is governed by

$$\mathcal{L}_1 = \sqrt{-g}\eta(R - \lambda) \quad (52)$$

where $\sqrt{-g}R$ coincides with $2\varepsilon^{\mu\nu}\partial_\mu\omega_\nu$, while the η in (52) coincides with η_2 in (51). [9], [10]

On the other hand if Lie algebra is chosen to be (48b), the gauge theoretical dynamics becomes equivalent to that of the recently much discussed, string-inspired model, with Lagrange density

$$\mathcal{L}_2 = \sqrt{-g}(\eta R - \lambda) \quad (53)$$

Note that in the gauge theoretical formulation based on the extended Poincaré group, the “cosmological constant” parameter λ does not appear in (51), even though it is present in (53). In fact λ arises as a solution to the gauge theoretic dynamics: one finds $\eta_3 = \lambda$, while η_2 continues to be identified with η in (53), and $\sqrt{-g}R$ remains $2\varepsilon^{\mu\nu}\partial_\mu\omega_\nu$. [11], [12]

Since quantization of the “ B - F ” gauge theory consists merely of solving its Gauss law, and this has been accomplished by formulas (31a) and (41), we conclude that quantization of the above two gravity theories (52) and (53) can also be completely and explicitly carried out.

Studying the quantum theory of these diffeomorphism invariant models, and also of the more complicated models, where matter degrees of freedom are coupled to the gravity

variables, promises to teach us valuable lessons about the nature of quantum gravity, albeit in the unphysical setting of lower dimensionality. The lower dimensionality precludes the existence of gravitons and their concomitant non-renormalizable interactions. But one retains the possibility of examining other questions about quantum gravity: the issue of quantizing a diffeomorphism invariant theory, the problem of time, the nature of the Wheeler-DeWitt equation, *etc.* [13]

NOTES

1. The momentum/curvature representation for gauge theories was introduced by Goldstone and Jackiw, and worked out in detail for $SU(2)$ in Ref. [1]. Generalization to other groups was given by Faddeev *et al.*, as well as Baluni and Grossman, Ref. [2]. Recent work on Yang-Mills theory in the curvature representation is by Freedman *et al.*, Ref. [3], while “ B - F ” theories, which arise in descriptions of gravity on a line, are discussed by Cangemi and Jackiw, Ref. [4], Amati *et al.*, Ref. [5] and Strobl *et al.*, Ref. [6].
2. Derivation of equation (12) within geometric quantization is due to V. P. Nair (unpublished).
3. An elementary discussion, together with references to the mathematical literature is in Bak *et al.*, Ref. [7].
4. For a discussion, see Jackiw in Ref. [8].

REFERENCES

- [1] J. Goldstone and R. Jackiw, Phys. Lett. **74B**, 81 (1979).
- [2] A. G. Izergin, V. E. Korepin, M. A. Semenov-Tyan-Shanskii and L. D. Faddeev, Teor. Mat. Fiz., **38**, 3 (1979) [Engl. trans.: Theor. Math. Phys. **38** 1 (1979).]; V. Baluni and B. Grossman, Phys. Lett. **78B**, 226 (1978); V. Baluni, *Phys. Lett.* **90B**, 407 (1980).
- [3] M. Bauer, D. Z. Freedman and P. E. Haagensen, Nucl. Phys. **B428**, 147 (1994); M. Bauer and D. Z. Freedman, Nucl. Phys. **B450**, 209 (1995); D. Z. Freedman, Nucl. Phys. B (Proc. Suppl.) **39B**, 477 (1995).
- [4] D. Cangemi and R. Jackiw, Phys. Rev. D **50**, 3913 (1994).
- [5] D. Amati, S. Elitzur and E. Rabinovici, Nucl. Phys. **B418**, 45 (1994).
- [6] T. Strobl, Phys. Rev. D **50**, 7346 (1994); A. Y. Alekseev, P. Schaller and T. Strobl, Phys. Rev. D **52**, 7146 (1995).
- [7] D. Bak, R. Jackiw and S.-Y. Pi, Phys. Rev. D **49**, 6778 (1994).
- [8] S. Treiman, R. Jackiw, B. Zumino and E. Witten, *Current Algebra and Anomalies*, (Princeton University Press/World Scientific, Princeton, NJ/Singapore, 1985).
- [9] R. Jackiw, C. Teitelboim in *Quantum Theory of Gravity*, S. Christensen, ed. (Adam Hilger, Bristol UK, 1984).
- [10] T. Fukuyama and K. Kamimura, Phys. Lett. B **160**, 259 (1985); K. Isler and C. Trugenberger, Phys. Rev. Lett. **63**, 834 (1989); A. Chamseddine and D. Wyler, Phys. Lett. **B228**, 75 (1989).
- [11] D. Cangemi and R. Jackiw, Phys. Rev. Lett. **69**, 233 (1992); Phys. Lett. **B299**, 24 (1993); Ann. Phys. (NY) 225, 229 (1993).
- [12] C. Callan, S. Giddings, J. Harvey and A. Strominger, Phys. Rev. D **45**, 1005 (1992); H. Verlinde in *Sixth Marcel Grossmann Meeting on General Relativity*, M. Sato and T. Nakamura, eds. (World Scientific, Singapore, 1992).

- [13] D. Cangemi and R. Jackiw, Phys. Rev. D **50**, 3913 (1994); Phys. Lett. **B337**, 271 (1994);
D. Cangemi, R. Jackiw and B. Zwiebach, Ann. Phys. (NY) **245**, 408 (1996); D. Bak
and D. Seminara, Phys. Rev. D **53**, 1907 (1996).